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PART IV] SECTION A [VOL. 17

ON THE RATIONAL SOLUTION OF THE DIOPHANTINE EQUATION $a(x^{m-1}-1)-b(y^{n-1}-1)=z^{p-1}$

By

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(Communicated by Professor A. C. Banerji, Head of the Department of Mathematics, Allahabad University, Allahabad)

Received on 24-3-49

Tchacaloff and Karamicoloff[1] have considered the solution of of the Diophantine equation $Ax^m+By^n=z^p$ where $(m, n)=1=m, p=(n, p)$. Siegal[2] has proved the possibility of the solution of the Diophantine equation $ax^n-by^n=C$.

The Diophantine equation $a(x^{m-1}-1)-b(y^{n-1}-1)=z^{p-1}$ is required in solving many problems in Theory of numbers. Here I shall prove the possibility of the existence of rational solution.

The following theorem is required:—

Theorem 1. If m, n, p are Coprime integers and $M=(m-1)(n-1)(p-1)$ *and* $M_1=\dfrac{M}{m-1}$, $M_2=\dfrac{M}{n-1}$, $M_3=\dfrac{M}{p-1}$ and α_1, α_2, α_3 be a set of integers such that $1+\alpha_r M_r\equiv 0 \ (mod\ mr-{}^{-}1)$ $mr=m, n, p$, then the rational solution of the Diophantine equation $Ax_1^{m-1}-Bx_2^{n-1}=x_3^{p-1}$ for which $x_1, x_1, x_3 \neq 0$ will be given by

$$x_1=X_1^{\frac{1+\alpha_1 M_1}{m-1}} X_2^{\frac{\alpha_2 M_2}{m-1}} X_3^{\frac{\alpha_3 M_3}{m-1}}$$

$$x_2=X_1^{\frac{\alpha_1 m_1}{n-1}} X_2^{\frac{1+\alpha_2 M_2}{n-1}} X_3^{\frac{\alpha_3 M_3}{n-1}}$$

$$x_3=X_1^{\frac{\alpha_1 M_1}{p-1}} X_2^{\frac{\alpha_2 M_2}{p-1}} X_3^{\frac{1+\alpha_3 M_3}{p-1}}$$

where $AX_1 - BX_2 = X_3$.

If α_1 be the value of α satisfying the equation $1+\alpha_1 M_1 \equiv 0$ (*mod* $m-1$) then $1+\alpha_1 M_1+\lambda(m-1) M_1 \equiv 0$ (*mod* $m-1$). Hence the equation $1+\alpha_1 M_1 \equiv 0$ (*mod* m_1-1) is satisfied in an infinity of ways.

It is evident that we can always find integers X_1, X_2, X_3 satisfying the equation $A X_1 - B X_2 = X_3$ when A and B are rational.

If X_1, X_2, X_3 satisfy the equation $AX_1 - BX_2 = X_3$ then $X_1^1 = X_1 + \lambda B, \times_2^1 = X_2 \times \lambda A$ will also satisfy the equation. Hence the equation $A X_1 - BX_2 = X_3$ may be solved in infinite ways.

Hence the Diophantine equation $A X_1^{m-1} BX_2^{n-1} = X_3^{p-1}$ may be solved in an infinite ways.

Since $1+M_r \alpha_r \equiv 0$ (*mod* $mr-1$) and $M_1 \equiv 0$ (*mod* $p-1, n$-1); $M_2 \equiv 0$ (*mod* m-1, $p-1$); $M_3 \equiv 0$ (*mod* $m-1$, $n-1$) and X_1, X_2, X_3 are integers then x_1, x_2, x_3 will also be integers.

Theorem 2. The Diophantine equation $a(x^{m-1}-1)-b(y^{n-1}-1) = z^{p-1}$ may be solved in integers in an infinite ways.

Let $x^{m-1} - \lambda x_1^{m-1} = 1$ *and* $y^{n-1} - \mu r x_2^{n-1}$, $x_3 = 2$, $a^1 = \lambda a$, $b^1 = \mu l$.

Then $a^1 x_1^{m-1} - b^1 x_2^{n-1} = x_3^{p-1}$ . . . . (1)

By theorem 1 the equation (1) may be solution integers in an infinite ways.

Again [31] the equation $x^{m-1} - \lambda x_1^{m-1} = 1$ has at most one solution in integers x, y when λ exceeds certain limit which depends on m when $m > C$.

Since we have infinite number of integral values of x_1 we have infinite number of integral values of x satisfying the equation $x^{m-1} - \lambda x_1^{m-1} = 1$. similarly for

Hence the equation $a(x^{m-1}-1) - b(y^{n-1}-1) = z^{p-1}$ may be solved in an infinite ways where m, n, p are relatively prime.

Reference.

1. Tchacaloff and Karamicoloff (1940) Comptes Rendus (Paris) p. 210
2. Siegel (1930) *Berliner Sitzungslerichte* pp. 1-70
3. Siegel. l.c.

NEW LINES IN CARBON SPECTRUM

By

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(Received on February 28, 1948)

(Communicated by Prof. D. B. Deodhar, Lucknow University, Lucknow.)

ABSTRACT

The emission spectra of iron and pure samples of carbon were photographed on a Hilger Medium size Quartz Spectrograph E 3 in the region of 2,100 to 7,000 A. U. by arcs running on 220 volts. D. C. and carrying current of 3 amps. The lines in the iron spectrogram were identified and the relative positions of the carbon lines were measured by Hilger's Comparator and their wave lengths were calculated by applying Hartmann's Interpolation formula. The lines due to impurities present in carbon specimens were eliminated, but in addition to the published carbon lines there was a positive indication of the presence of a fairly large number of new lines of varying intensity which are attributed to carbon.

Introduction

Much work has been published in the past years on the analysis of line and band spectra of carbon in the ultra-violet and the extreme ultra violet regions by Fowler and Selwyne (1928) and Johnson (1925)

In the course of the spectroscopic investigations on some compounds with the help of the carbon arc it was noticed that there were some new lines attributable to carbon, and it was thought desirable to make an independent observation on the carbon arc to settle this point. A carbon arc of spectroscopically pure specimens of carbon rods was set up for this purpose and spectrograms were photographed in the region of 2,100 to 7,000 A. U.

The observations were repeated a number of times and it was discovered that new lines existed on all the plates. These lines do not seem to have been recorded by previous workers.

EXPERIMENTAL.

The Hilger medium size quartz spectrograph was used to photograph the iron and carbon arcs in juxtaposition with two consecutive holes of Hartmann's diaphragm. The essential feature of this spectrograph is the Cornu Prism which is composed of two 30° prisms of quartz of opposite rotatory power-one dextro and the other lævo rotatory prisms in optical contact so as to cancel the optical rotations.

Ilford Hypersensitive Panchromatic Plates H P 3 were exposed to photograph the emission spectra of the arcs run on 220 volts D. C. carrying 3 amps. current. These plates were exposed and developed in absolute darkness and the spectrogram was phtographed in the region extending from 2,100 to 7,000 A. U.

The lines of the iron spectrogram were identified and compared with the standard spectrograms published by Hilgers and the positions of the carbon lines were measured with reference to the iron lines of known wavelengths by means of the Hilger measuring micrometer L 13 and their wave-lengths were calculated by appyling Hartmann's formula which is fairly accurate over short ranges of wave-lengths.

The Hartmann's interpolation formula gives

$$\lambda = \lambda_o + \frac{C}{\eta - \eta_o}$$

where

λ is the wave-length of the line in question

λ_o, C, η_o are constants

and η, the position of the unknown line on tne micrometer scale.

Measuring the distances of three known lines of Fe-spectrum and substituting in the above formula provides a set of simultaneous equations from which all the three unknown constants can be easily evaluated.

Knowing all these constants and measuring the distance of the unknown carbon line with reference to the identified iron line the wave-length of the unknown line can be calculated.

In this way the wave-lengths of a large number of carbon lines have been measured.

Observations

The carbon electrodes used for the arc spectrum were supplied by Adam Hilger Ltd. London.

The report accompanying these carbon electrodes which were prepared under extraordinary special conditions with a view to obtain purest possible carbon revealed the presence of the following impurities in traces on spectrographic analysis

Sodium	Na
Magnesium	Mg
Calcium	Ca
Strontium	Sr
Copper	Cu
Manganese	Mn
Aluminium	Al
Silicon	Si

According to the report, the lines due to Potassium Barium and Aluminium were not in evidence and the report provides a table of wave-lengths of th lines due to impurities present in the specimen.

These lines have been completely eliminated from the spectrogram of the carbon, photographed. Further all those lines found in the carbon spectrum appended in standard tables (1924) and (1932) have been taken into consideration. Apart from all these lines, there is a positive evidence of the presence of some more lines in the spectrum which have not been published by other workers.

A table of the new carbon lines as observed is given below :—

S. N.	Wave-length of new Carbon lines	Frequency	Intensity
	A. U.	cm^{-1}	
1	4197.600	23823.13	3
2	3851.727	25962.39	3
3	3202.585	31224.79	2
4	3197.859	31270.93	2
5	3191.122	31336.96	1
6	3185.438	31392.87	4
7	3183.917	31407.86	5
8	3133.224	31917.10	3
9	3130.246	31947.37	2
10	3125.591	31993.96	3
11	3120.705	32044.05	2
12	3118.442	32067.3	5
13	3053.327	32751.17	4
14	3033.426	32966.03	2
15	2957.261	33815.08	1
16	2941.285	33998.75	2
17	2920.066	34245.8	2

S. N.	Wave-length of new Carbon lines A. U.	Frequency cm^{-1}	Intensity
18	2906.113	34410.2	2
19	2892.510	34572.05	1
20	2701.375	37018.18	2
21	2690.844	37163.06	2
22	2683.423	37265.84	2
23	2388.59	41865.70	4
24	2375.18	42102.08	5

The intesity of these lines has been recorded taking the intensity of the brightest line in the spectrogram as 10

Conclusions

After eliminating all lines due to impurities in the carbon specimen and the published lines of carbon there is a positive indication of the existence of 24 additional lines of carbon, the wave-lengths, frequencies and intensities of which are given in a tabular form.

A major portion of these lines are faint in intensity but their positions can be properly located with the help of the Hilgers Measuring micrometer L 13 using a suitable magnification.

These lines do not appear to belong to the bands present in the carbon spectrum

Acknowledgement

In conclusion, the author expresses his sincere and grateful thanks to Prof. D. B. Deodhar for suggesting the problem and for his valuable guidance and to the University for a research grant.

Department of Physics Lucknow University.
Dated 24 th Feb 1948.

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STRAZZERI'S FORMULA IN RECTILINEAR CONGRUENCES AND ITS APPLICATIONS

By

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The object of this paper is to establish Strazzeri's formula * in Rectilinear Congruences by tensor methods and to obtain from it or directly various forms for $\sin\theta$ where θ is the angle which a ray of the congruence makes with the normal to the surface of reference at its point of intersection with the surface.

1. Let a Rectilinear Congruence be defined by the co-ordinates $x^i = x^i(u^1, u^2)$, $i=1, 2, 3$ of a point M on the surface of reference S and by the direction cosines $\lambda^i = \lambda^i(u^1, u^2)$, $i=1, 2, 3$ of the line passing through M.

Then $$\lambda^i . \lambda^i = 1. \qquad \dots \quad (1{\cdot}1)$$

The functions λ^i may be expressed in terms of the direction numbers $x^i,_{\alpha}$ $(\alpha=1, 2)$ of the tangents to the co-ordinate curves on the surface through M, and the direction cosines X^i of the normal to the surface at M. Thus

$$\lambda^i = p^\alpha x^i,_{\alpha} + qX^i \qquad \dots \quad (1{\cdot}2)$$

where p^α are the contravariant components of a vector in the surface at M, 'q' is a positive scalar function and $x^i,_{\alpha}$ denotes covariant differentiation of x^i with regard to u^α based on the first fundamental tensor

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* See Strazzeri, Palermo Rendiconti (1927), p. 138 ; Behari, R, Jour. Ind. Math. Soc., New Series, Vol. II (1936), pp. 163-164.

$$g_{\alpha\beta}=x^i_{,\alpha}x^i_{;\beta} \quad \ldots \quad (1\cdot3)$$

of the surface S.

From (1·2) we get, by virtue of (1·1) and (1·3),

$$\lambda^i.\lambda^i\equiv(p^\alpha x^i_{,\alpha}+qX^i)(p^\beta x^i_{,\beta}+qX^i)=p^\alpha p^\beta g_{\alpha\beta}+q^2$$

$$\text{or } 1=p_\alpha p^\alpha+q^2=p_\alpha p_\beta g^{\alpha\beta}+q^2 \quad \ldots \quad (1\cdot4)$$

If θ is the angle between the normal to 'S' at M and the line λ of the congruence at M, it follows from (1·2) that

$$\cos\theta=\lambda^i.X^i=q \quad \ldots \quad (1\cdot5)$$

The equation (1·4) assumes the form

$$p_\alpha\, p_\beta g^{\alpha\beta}=\sin^2\theta \quad \ldots \quad (1\cdot6)$$

Suppose the determinant $(x^i_{,\alpha}x^i_{;\beta}\lambda^i)=e_{\alpha\beta}$ ... (1·7)

then

$$e_{\alpha\beta}\, e_{\gamma\delta}=g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}\, g_{\beta\gamma} \quad \ldots \quad (1\cdot8)$$

and the equation (1·6) becomes

$$p_\alpha p_\beta\, g_{\delta\gamma}e^{\alpha\gamma}\, e^{\beta\delta}=\sin^2\theta \quad \ldots \quad (1\cdot9)$$

2. The direction numbers $x^i_{,\alpha}$ $(\alpha=1,2)$ of the tangents to the co-ordinate curves on the surface of reference through M, may be expressed in terms of λ^i and $\lambda^i_{;\alpha}$ $(\alpha=1,2)$. Thus

$$x^i_{,\alpha}=p_\alpha\lambda^i+q^\gamma_{\ \alpha}\lambda^i_{,\gamma} \quad \ldots \quad (2\cdot1)$$

where $\lambda^i_{,\alpha}$ denotes covariant differentiation of λ^i with regard to u^α based on the first fundamental tensor of the spherical representation of the congruence,

$$G_{\alpha\beta}=\lambda^i_{;\alpha}.\lambda^i_{;\beta} \quad \ldots \quad (2\cdot2)$$

From (2·1)

$$\lambda^i.x^i_{,\alpha}=p_\alpha \quad \ldots \quad (2\cdot3)$$

$$q^\gamma_{\ \alpha}=x^i_{,\alpha}x^i_{,\delta}\, g_{\beta\phi}\, E^{\gamma\beta}\, E^{\delta\phi} \quad \ldots \quad (2\cdot4)$$

$$\text{where } E_{\alpha\beta}=(\lambda^i_{,\alpha}\lambda^i_{,\beta}\lambda^i), \quad \ldots \quad (2\cdot5)$$

$$E_{\alpha\beta}\, E_{\gamma\delta}=G_{\alpha\gamma}\, G_{\beta\delta}-G_{\alpha\delta}\, G_{\beta\gamma} \quad \ldots \quad (2\cdot6)$$

and $E^{\alpha\beta}$ is a tensor conjugate to $E_{\alpha\beta}$.

The first fundamental tensor of the surface of reference S is given by

$$g_{\alpha\beta} \equiv x^i{}_{,\alpha} \cdot x^i{}_{,\beta} = p_\alpha p_\beta + q_\alpha{}^\gamma q_{\gamma\beta}$$

whence, using the equations (2·4), we get using Eisenhart's notation

$$\left.\begin{aligned} E &\equiv g_{11} = p_1{}^2 + q_1{}^\gamma q_{\gamma 1} = p_1{}^2 + \frac{1}{H^2}[\mathrm{G}e^2 - 2\mathrm{F}ef' + \mathrm{E}f'^2] \\ F &\equiv g_{12} = p_1 p_2 + q_1{}^\gamma q_{\gamma 2} = p_1 p_2 + \frac{1}{H^2}[\mathrm{G}ef - \mathrm{F}ff' - \mathrm{F}eg + \mathrm{E}gf'] \\ G &\equiv g_{22} = p_2{}^2 + q^\gamma{}_2 q_{\gamma 2} = p_2{}^2 + \frac{1}{H^2}[\mathrm{G}f^2 - 2\mathrm{F}gf + \mathrm{E}g^2] \end{aligned}\right\} \quad \ldots \quad (2·7)$$

From these equations

$$[e_{\alpha\beta} E^{\alpha\delta}]^2 = [q^\alpha{}_\beta\, q^\gamma{}_\delta\, E_{\alpha\gamma}\, E^{\beta\delta}]^2 + [e_{\alpha\beta} E^{\alpha\delta}]^2 p_\alpha p_\beta g_{\delta\varphi} e^{\alpha\delta} \epsilon^{\beta\phi}$$

or using the equations (1·8) and (2·4) in this equation we get

$$\frac{EG-F^2}{\mathrm{EG}-\mathrm{F}^2}\cos^2\theta = \frac{(eg-ff')^2}{(\mathrm{EG}-\mathrm{F}^2)^2} \text{ or } \frac{ds}{d\sigma}\cos\theta = \rho_1\rho_2 \quad \ldots \quad (2·8)$$

where ds and $d\sigma$ are the elements of areas of the surface of reference at M and the spherical representation of the congruence and ρ_1, ρ_2 are the distances to the focal points from the surface of reference.

This formula can be proved alternately by considering the determinant

$$(x^i{}_{,\alpha}\; x^i{}_{,\beta}\; \lambda^i),$$

which with the help of the equation (1·2) can be written as

$$(x^i{}_{,\alpha}\; x^i{}_{,\beta}\;\; p^\alpha x^i{}_{,\alpha} + qX^i).$$

In virtue of (1·7) this determinant becomes equal to

$$q e_{\alpha\beta}.$$

using (2·1) the determinant also beomes

$$(p_\alpha \lambda^i + q^\gamma{}_\alpha \lambda^i{}_{,\gamma}\;\; p_\beta \lambda^i + q^\delta_\beta \lambda^i{}_{,\delta}\;\; \lambda^i)$$

which using (2·5) is equal to ... (2·9)

$$q^\gamma{}_\alpha q^\delta_\beta E_{\gamma\delta}.$$

Hence $$q e_{\alpha\beta} = q^{\gamma}{}_{\alpha} q^{\delta}{}_{\beta} E_{\gamma\delta}$$
or
$$q e_{12} = (q^1{}_1 q^2{}_2 - q^2{}_1 q^1{}_2) E_{12}$$
or using (2·4) we get $q \dfrac{e_{12}}{E_{12}} = \dfrac{eg - ff'}{EG - F^2}$

Hence $$\cos\theta . \frac{ds}{d\sigma} = \rho_1 \rho_2 .$$

3. The equation (1·9) gives an expression for $\sin^2\theta$ in terms of p_α,
$$\sin^2\theta = p_\alpha p_\beta \, g_{\delta\gamma} e^{\alpha\gamma} e^{\beta\delta}.$$

When expanded this equation becomes
$$\sin^2\theta = \frac{1}{H^2}\left[p_1^2 G + p_2^2 E - 2 p_1 p_2 F\right] \qquad \ldots \qquad (3\cdot1)$$

Using Strazzeri's formula we now find out another expression for $\sin^2\theta$.

Covariant differentiation of λ^i in the equation (1·2) gives
$$\lambda^i{}_{,\beta} = p^\alpha x^i{}_{,\alpha\beta} + p^\alpha{}_{,\beta} x^i{}_{,\alpha} + q X^i{}_{,\beta} + X^i q_{,\beta}$$
which by means of Gauss and Weingarten equations
$$\left.\begin{aligned} x^i{}_{,\alpha\beta} &= d_{\alpha\beta} X^i \\ \text{and} \quad X^i{}_{,\beta} &= -d_{\beta\gamma} g^{\gamma\delta} x^i{}_{,\delta} \end{aligned}\right\} \qquad \ldots \qquad (3\cdot2)$$
can be written in the form
$$\lambda^i{}_{,\beta} = \mu^{\gamma}_{\beta} x^i{}_{,\gamma} + \nu_\beta X^i$$
$$\text{where } \mu^\gamma{}_\alpha \equiv p^\gamma{}_{,\alpha} - q d_{\alpha\sigma} g^{\sigma\gamma}$$
$$\nu_\alpha \equiv q_{,\alpha} + p^\beta d_{\alpha\beta} \qquad \ldots \qquad (3\cdot3)$$

From the equation (3·3) the first fundamental tensor for the spherical representation of the congruence is given by
$$G_{\alpha\beta} \equiv \lambda^i{}_{,\alpha} \lambda^i{}_{,\beta} = \mu^{\gamma}_{\beta} \mu^{\delta}_{\alpha} g_{\gamma\delta} + \nu_\alpha \nu_\beta \qquad \ldots \qquad (3\cdot4)$$

Also using equation (3·3)

$$\lambda^i{}_{\beta} x^i{}_{;\alpha} = \mu^\gamma{}_\beta\, g_{\alpha\gamma}$$

or $\mu^\gamma{}_\beta = g^{\gamma\alpha}\, \lambda^i{}_{;\beta} x^i{}_{;\alpha} = g_{\delta\varphi}\, e^{\alpha\delta}\, e^{\varphi\gamma}\, \lambda^i{}_{;\beta} \cdot x^i{}_{;\alpha}$

whence $\mu^1{}_1 = \dfrac{Gs-Ff'}{H^2}$, $\mu^2{}_1 = \dfrac{Ef'-Fe}{H^2}$, $\mu^1{}_2 = \dfrac{Gf-Fg}{H^2}$, $\mu^2{}_2 = \dfrac{Eg-Ff}{H^2}$ ...(3′5)

Using (3·5) in (3·4) we get,

$$\left.\begin{aligned} E &= G_{11} = \nu^2{}_1 + \mu^\gamma{}_1\, \mu_{1\gamma} = \nu_1{}^2 + \frac{1}{H^2}\,[Ge^2 - 2\,Fef' + Ef'^2] \\ F &= G_{12} = \nu_1\nu_2 + \mu^\gamma{}_1\, \mu_{2\gamma} = \nu_1\nu_2 + \frac{1}{H^2}[Gef - Fff' - Feg + Egf'] \\ G &= G_{22} = \nu_2{}^2 + \mu_2{}^\gamma\, \mu_{2\gamma} = \nu_2{}^2 + \frac{1}{H^2}\,[Gf^2 - 2\,Fgf + Eg^2] \end{aligned}\right\} \quad \ldots \quad (3{\cdot}6)$$

From these equations

$[E_{\alpha\beta} e^{\alpha\beta}]^2\,(1 - \nu_\alpha\nu_\beta\, G_{\gamma\delta}\, E^{\alpha\gamma} E^{\beta\delta}) = (q^\alpha{}_\beta\, q^\delta{}_\gamma\, e_{\alpha\delta}\, e_{\beta\gamma})^2\,(E_{\alpha\beta}\, e^{\alpha\beta})^4$

or $(e_{\alpha\beta}\, E^{\alpha\beta})^2\,(1 - \nu_\alpha\nu_\beta\, G_{\gamma\delta}\, E^{\alpha\gamma}\, E^{\beta\delta}) = (q^\alpha{}_\beta\, q^\delta{}_\gamma\, E_{\alpha\delta} E_{\beta\gamma})^2$

or $\dfrac{EG-F^2}{\mathrm{EG}-\mathrm{F}^2}(1 - \nu_\alpha\nu_\beta\, G_{\gamma\delta}\, E^{\alpha\gamma} E^{\beta\delta}) = \left(\dfrac{eg-ff'}{\mathrm{EG}-\mathrm{F}^2}\right)^2$

or $\left(\dfrac{ds}{d\sigma}\right)^2 (1 - \nu_\alpha\nu_\beta\, G_{\gamma\delta}\, E^{\alpha\gamma}\, E^{\beta\delta}) = (\rho_1\rho_2)^2$

$$\therefore \qquad \sin^2\theta = \nu_\alpha\nu_\beta\, G_{\gamma\delta}\, E^{\alpha\gamma}\, E^{\beta\delta} \qquad \ldots \quad (3{\cdot}7)$$

in consequence of Strazzeri's formula.

When expanded this expression assumes the form

$$\sin^2\theta = \frac{\nu_1{}^2\mathrm{G} + \nu_2{}^2\mathrm{E} - 2\nu_1\nu_2\mathrm{F}}{\mathrm{H}^2} \qquad \ldots \quad (3{\cdot}8)$$

4. The expression (3·7) for $\sin^2\theta$ can be obtained independently of Strazzeri's formula as follows:—

The functions X^i may be expressed in terms of λ^i and $\lambda^i{}_{;\alpha}$.

Thus $$X^i = \nu^\alpha . \lambda^i{}_{;\alpha} + q\lambda^i \qquad \ldots \quad (4{\cdot}1)$$

where ν^α are the contravariant components of a vector in the spherical representation of the congruence, 'q' is a positive scalar function and

$\lambda^i_{,\alpha}$ denotes covariant differentiation of λ^i with regard to u^α based on the first fundamental tensor

$$G_{\alpha\beta}=\lambda^i_{,\alpha}.\ \lambda^i_{,\beta} \qquad \ldots \quad (4\cdot2)$$

of the surface S.

From the equation (4·2), $\cos\theta \equiv \lambda^i X^i = q$,

$$X^i \lambda^i_{,\alpha} = \nu^\beta G_{\alpha\beta} = \nu_\alpha$$

and
$$X^i . X^i = (\nu^\alpha \lambda^i_{;\alpha} + q\lambda^i)(\nu^\beta \lambda^i_{\beta} + q\lambda^i)$$

or
$$1 = \nu^\alpha \nu^\beta G_{\alpha\beta} + q^2$$

or
$$\sin^2\theta = \nu_\alpha \nu^\alpha$$

or
$$\sin^2\theta = \nu_\alpha \nu_\beta G^{\alpha\beta}$$

or
$$\sin^2\theta = \nu_\alpha \nu_\beta E^{\alpha\gamma} E^{\beta\delta} G_{\gamma\delta}$$

in virtue of (2·5) and (2·6).

NOTE ON CONICS OF DOUBLE OSCULATION OF A CUBIC

By

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ABSTRACT

As its title indicates, this paper (divided into four sections) is devoted to a somewhat detailed discussion of the nine systems of (∞^1 of) conics of double osculation—real or imaginary—attaching respectively to the nine points of inflexion of a given (bicursal) cubic Γ.

In Sec. I Chasles's characteristics, associated with any of the nine sets of conics have been determined, and the loci of the centres and foci have been shewn to be of respective degrees 4 and 12, whereas the envelope of the asymptotes is found to be of class 14.

Notice has been further taken of the

(i) *three* rectangular hyperbolas; (ii) *four* parabolas;

(iii) *four* pairs of lines (inflexional tangents);

and (iv) *three* 'cognate' sextactic conics,

that are included in any of the nine aforesaid systems of conic of double osculation. Incidentally, a symbolic (irrational) form of Γ in terms of three 'cognate' conics of double osculation has been established.

In Sec. II the M-invariant, the Jacobian (covariant) and a certain relevant contravariant, related to an arbitrary triad of 'cognate' conics of double osculation, have been taken into consideration. Special attention has been paid to the triad of (cognate) conics (of double osculation), qualified by a *zero* M-invariant.

In point of fact, it has been definitely proved that such a triad of conics—which includes, as a particular variety, the three 'cognate' sextactic conics—possesses a common self-conjugate triangle.

A48-4-3

Sec. III deals principally with the canonical *homogeneous* forms, which three 'cognate' conics of double osculation—having a *null* M-invariant—assume, when their common self-conjugate triangle is chosen as the *triangle of reference*. Prominence has been given to the particular case when the three ('cognate') conics in question happen to be the three sextactic conics, included in the system.

Lastly, Sec. IV begins with a lemma, touching on the *infinitude* of 'equi-anharmonic' cubics, 'each of which counts, among its aggregate of polar conics', three *assigned* conics, endowed with a common self-conjugate triangle. The lemma has been finally applied to the investigation of the series of 'equi-anharmonic' cubics, that have for their common polar conics, an assigned triad of 'cognate' conics of double osculation,—which may or may not be sextactic conics,—related to a given (bicursal) cubic and qualified by a *zero* M-invariant.

INTRODUCTION

The present paper, as its name implies, is devoted to a discussion on the nine systems of conics of double osculation (real or imaginary), which a bicursal cubic possesses under normal circumstances. As is well-known, (non-degenerate) plane cubics are classified as *bicursal* or *unicursal*, according as their *deficiency* or *genus* is 1 or 0. Little or no notice has been taken in this paper of the comparatively simple theory of *unicursal* cubics, considering that they are derivable from ordinary conics by the process of "quadric inversion" (i. e., circular inversion combined with projection). Of course, most of the properties of *bicursal* cubics, when *properly* modified, hold also for *unicursal* varieties.

As regards the *special* use of the term *osculation* in this paper, a few words of explanation are called for. It is common knowledge that, whilst the 'osculating conic' of a curve Γ at a point P is generally defined as the conic, having five-pointic contact with Γ near P, two arbitrary curves Γ and Γ' are, generally speaking, said to '*osculate*' each other at a point P, provided they have only *three-pointic* (and *not* five-pointic) contact with each other near P. In the present context, by a '*conic of double osculation*' of a cubic Γ is meant a conic, having *three pointic* contacts with Γ at *two* points, which are usually *distinct* but may in special cases coincide.

As a matter of convenience, the paper has been divided into four sections. Section I treats of certain characteristic properties of any one of the *nine* systems of ∞^1 conics of double osculation, attaching respectively to the *nine* points of inflexion of the cubic Γ. The infinitude of conics of any of the *nine* systems—spoken of as 'cognate' to one another—contains, among others, three *sextactic* conics of Γ. Then Sec. II deals with the M-invariant, the Jacobian covariant and cubic contravariant, associated with an *arbitrary triad* of 'cognate' conics of double osculation and rightly lays great stress on a *specialised* triad of such conics, *having a zero M-invariant*; the intrinsic importance of the last-named category of triads (of conics of double osculation) lies in the fact that it includes, within its fold, the triad of '*sextactic*' conics. Thirdly Sec. III disposes of the *canonical* (homogeneous) forms of a triad of '*cognate*' conics (of double osculation)—belonging to any one of the *nine* systems—having a zero M-invariant. Finally Sec. IV begins with a digression on an arbitrary triad of conics, having a common self-conjugate triangle, and ends with an application to the special triads of conics of double osculation (of a cubic), having a *zero* M-invariant.

I beg leave to add that, although the subject-matter of this paper is classical in origin and there are occasions on which I have felt constrained to touch on *known* results, still I honestly believe that this paper embodies a very decent amount of original contributions to the subject. Even in the disposal of known results, there is, I believe, some *novelty* in the mode of treatment.

SECTION I

(General properties of systems of conics of double osculation.)

Art. 1.—We know from the Theory of Higher Plane Curves

(i) that if a conic has three-pointic contacts with a given cubic Γ (bicursal or unicursal) at two points P, Q, the chord PQ must cut Γ at a point of inflexion I;

and (ii) that, conversely, if an arbitrary transversal IPQ be drawn through a point of inflexion I of the cubic Γ, then a uniquely determinate conic can be described so as to have three-pointic contacts with Γ at P, Q.

Thus, intimately related to a point of inflexion I of a cubic Γ, there exists a family of ∞^1 conics of double osculation, whose chords of (double) osculation form a *pencil* through I. If, then, Γ be supposed to be a *bicursal* cubic of the most unrestricted type, it must have *nine* sets of ∞^1 conics of double osculation, attaching respectively to the *nine* points of inflexion (real or imaginary). For felicity of expression, we shall use the adjunct "cognate", in connection with two or more conics (of double osculation), to signify that the conics in question belong to the *same* family, *i.e.*, that their associated chords of double osculation pass through the *same* point of inflexion of the cubic. In other words, two conics of double osculation (of Γ) will be termed "cognate" or "non-cognate", according as their related chords (of double osculation) do or do not pass through the *same* point of inflexion. Adopting this nomenclature, we can simply assert that a bicursal cubic admits, in general, of nine distinct families of *cognate* conics of double osculation. The main purpose of this paper is to make a systematical study of the afore-mentioned families of conics. Our object will be realised if we confine our attention to *only one* of the *nine* systems of conics (of double osculation); for the intrinsic properties, shewn to hold for *any one* of these systems, must, from considerations of symmetry, hold also for each of the *remaining eight* systems.

In the analytic investigation to be initiated in the next article we shall find it to our convenience to approach the subject in an *indirect* manner.

Art. 2.—We shall first make use of a lemma, (the truth of which is intuitively evident) viz.,

If P, Q be the two points (real or imaginary), where a *given* conic S is met by a *given* right line u, the *most general* equation of a cubic, having three-pointic contacts with S at each of points P, Q is representable in the form :

$$S.\,v=u^3, \qquad \ldots \quad \text{(I)}$$

where v is an *arbitrary* right line, involving necessarily *three* arbitrary (or disposable) constants.

A cursory glance at (I) suggests that the chord of double osculation, viz., u cuts Γ at an inflexion, the tangent at which is v,—a fact otherwise obvious on *a priori* grounds. We may now look at the matter from a different perspective, and affirm that, if the cubic be *given* in the first instance and S be any of the infinitude of conics of double osculation, the equation of Γ must be of the symbolic form (I), where, of course, u, v are two right lines, of which the former is the chord of osculation, and the latter is the tangent (to Γ) at the related point of inflexion. If, as a measure of expediency, this point of inflexion be taken as the origin O of Cartesian coordinates (in general, oblique), and the lines

$$u=0 \text{ and } v=0$$

be taken respectively as the coordinate axes

$$x=0 \text{ and } y=0,$$

the Cartesian equation of the original bicursal cubic Γ may be thrown into the form

$$S.\,y=x^3, \qquad \ldots \quad \text{(II)}$$

where the conic, viz.,

$$S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad . \quad . \quad . \quad (1)$$

osculates Γ at its two points of intersection with the axis $x=0$. Needless to say, the axis $y=0$ is the tangent to Γ at the point of inflexion O.

The equation (II) being equivalent to

$$(S+3\lambda x^2+3\lambda^2 xy+\lambda^3 y^2).\, y=(x+\lambda y)^3, \qquad . \quad . \quad . \quad \text{(III)}$$

(where λ is an *arbitrary* parameter), we readily recognise that the *general* equation of the family of ∞^1 cognate conics of double osculation (including the first conic S as a particular member) is

$$\left.\begin{array}{l} S+3\lambda x^2+3\lambda^2 xy+\lambda^3 y^2=0, \\ \text{i.e., } (a+3\lambda)x^2+2(h+\tfrac{3}{2}\lambda^2)xy+(b+\lambda^3)y^2+2gx+2fy+c=0. \end{array}\right\} \quad . \quad . \quad . \quad (2)$$

It goes without saying that the pencil of corresponding chords of double osculation,—passing, of course, through the related point of inflexion O—is denoted by

$$x+\lambda y=0, \qquad . \quad . \quad . \quad (3)$$

where λ is, as before, a variable parameter.

It is clear on all hands that, by choosing the parameter λ in diverse ways, one can easily deduce, from (2), the equations of

(i) the *three* sextactic conics,

(ii) the *four* parabolas,

(iii) the *three* rectangular hyperbolas,

and (iv) the *four* pairs of right lines,

that are included in the family of conics of double osculation, appertaining to the point of inflexion O. Judging from elementary considerations, one can readily perceive that, unless the cubic Γ is *specialised* to a certain extent, the family will *not* include even a single circle.

The four cases (i), (ii), (iii) and (iv) will now be dealt with serially.

Case i. Observing that (2) will be a sextactic conic of Γ, if it touches the chord of (double) osculation viz. (3), we promptly realise that the *three* sextactic conics*, belonging to the system (2), are represented by the Cartesian equation

$$(a+3\lambda_r)\,x^2+2\,(h+\tfrac{3}{2}\lambda_r^2)\,xy+(b+\lambda_r^3)\,y^2+2gx+2fy+c=0, \qquad \text{(IV)}$$

and that the three associated sextactic points are

$$\left(\frac{c\lambda_r}{f-\lambda_r g},\ -\frac{c}{f-\lambda_r g}\right),$$

and that the tangents at these points are

$$x+\lambda_r y=0,$$

provided that r runs through the values 1, 2, 3 in succession, and that $\lambda_1, \lambda_2, \lambda_3$ are the three roots of the cubic+ in λ, viz.,

$$c\lambda^3+B\lambda^2+2H\lambda+A=0.\ldots \quad \ldots \quad \ldots \quad \ldots \qquad \text{(V)}$$

Case ii. The system (2) includes *four* parabolas, whose Cartesian equations are of the form:

$$(a+3k_r)x^2+2(h+\tfrac{3}{2}k_r^2)xy+(b+k_r^3)y^2+2gx+2fy+c=0, \qquad \text{(VI)}$$

provided that r runs through the values 1, 2, 3, 4, and that k_1, k_2, k_3, k_4 are the four roots of the biquadratic in k viz.,

$$(a+3k)\,(b+k^3)=(h+\tfrac{3}{2}k^2)^2.$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \qquad \text{(VII)}$$

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* The three sextactic conics will be hereafter referred to as a triad of "*cognate*" sextactic conics, and their related sextactic points will be termed "*cognate*". That is to say, two sextactic points are *cognate* or *non-cognate* according as they are or are not *co-tangential*. So the adjunct "cognate", when applied to sextactic points is synonymous with *co-tangential*, the common tangential being of course a point of inflexion of the cubic. Thus a bicursal cubic Γ must be said to possess nine triads of '*cognate*' sextactic conics and nine associated triads of "*cognate*" sextactic points.

+As usual, the capital letters (A, B, C, F, G, H) denote the co-factors of the corresponding small letters (a, b, c, f, g, h) in the determinant:

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Case iii. The system (2) includes *three* rectangular hyperbolas, whose Cartesian equations are of the type:

$$(a+3m_r)x^2+2(h+\tfrac{3}{2}m_r^2)xy+(b+m_r^3)y^2+2gx+2fy+c=0, \qquad \text{(VIII)}$$

provided that r runs through the values 1, 2, 3 and that m_1, m_2, m_3, are the three roots of the cubic in m, viz.,

$$(a+3m)+(b+m^3)-2(h+\tfrac{3}{2}m^2)\cos w=0, \quad . \ . \ . \quad \text{(IX)}$$

(w denoting the obliquity of the Cartesian axes).

Case iv. The system (2) includes *four* improper conics (i. e., conics composed of pairs of right lines), whose equations are of the form:

$$(a+3n_r)x^2+2(h+\tfrac{3}{2}n_r^2)xy+(b+n_r^3)y^2+2gx+2fy+c=0, \qquad \text{(X)}$$

provided that r runs through the values 1, 2, 3, 4, and that n_1, n_2, n_3, n_4 are the roots of the biquadratic in n, viz.,

$$\begin{vmatrix} a+3n, & h+\tfrac{3}{2}n^2, & g \\ h+\tfrac{3}{2}n^2, & b+n^3, & f \\ g\ , & f\ , & c \end{vmatrix} = 0. \quad . \ . \ . \quad \text{(XI)}$$

If (ξ_r, η_r) be the two constituent lines of the *improper* conic (X) for any of the four values of n_r, satisfying the equation (XI), and if we set

$$\zeta_r = x+n_r . y,$$

then the equation (II) or (III) of the original cubic Γ can be alternatively put in each of the four forms, viz.,

$$\xi_r \eta_r y = \zeta_r^3, \quad (r=1,2,3,4). \quad . \ . \ . \quad \text{(XII)}.$$

Remarking that the four lines of the type $\{\zeta_r\}$ all pass through the inflexion O, and that the related harmonic polar (say, ζ), viz.,

$$gx+fy+c=0$$

is the *common* polar line of O *w. r. t.* the different conics of the system (2),—including, as special varieties, the four line-pairs of the type $\{\xi_r, \eta_r\}$—and interpreting (XII) geometrically, we are squarely led to an *alternative* proof of the following known result:—

If O be any one of the nine points of inflexion (real or imaginary) of a bicursal cubic Γ, the remaining eight points of inflexion can be grouped into

four pairs, such that the two inflexions of each pair are collinear with O, and that their related tangents—of the type $\{\xi_r, \eta_r\}$*—intersect somewhere on the harmonic polar* ζ *of* 0.

Art 3. For the sake of brevity we shall now introduce the following notations:—

$$\left.\begin{aligned} S_\lambda &\equiv S+3\lambda x^2+3\lambda^2 xy+\lambda^3 y^2 \\ &\equiv (a+3\lambda)\, x^2+2\,(h+\tfrac{3}{2}\lambda^2)\, xy+(b+\lambda^3)\, y^2+2g\,x+2fy+c \end{aligned}\right\} \quad \ldots \quad \text{(I)}$$

and $T_\lambda \equiv x+\lambda y$. ... ... (II)

Accordingly, the equation of the original cubic Γ, viz.,

$$S.\,y = x^3 \quad \ldots \quad \ldots \quad \text{(III)}$$

can be alternatively put in the form

$$S_\lambda.\,y = T_\lambda^3. \quad \ldots \quad \ldots \quad \ldots \quad \text{(IV)}$$

It is hardly necessary to mention that, interpreted geometrically, the equation

$$S_\lambda = 0 \ldots \quad \ldots \quad \ldots \quad \text{(V)}$$

defines, for varying values of the parameter λ, a family of ∞^1 conics of double osculation, whilst the equation

$$T_\lambda = 0 \quad \ldots \quad \ldots \quad \text{(VI)}$$

defines the pencil of corresponding chords of (double) osculation, (passing, of course, through the point of inflexion O)

If we now select three *arbitrary* conics of the system (V) by ascribing three *arbitrary* values (say, λ_1, λ_2, λ_3) to λ, we can re-write (III) in each of the three symbolic forms:

$$S_{\lambda r}.\,y = (x+\lambda_r y)^3, \quad (r=1, 2, 3). \quad \ldots \quad \text{(VII)}$$

If, then, l, m, n denote the constants

$$\lambda_2-\lambda_3, \quad \lambda_3-\lambda_1, \quad \lambda_1-\lambda_2$$

respectively, the three equations of the type (VII) are easily seen to lead to the following irrational form (of Γ), viz.,

$$l\, S_{\lambda_1}^{\frac{1}{3}}+m\,.\,S_{\lambda_2}^{\frac{1}{3}}+n\,S_{\lambda_3}^{\frac{1}{3}}=0. \quad \ldots \quad \text{(VIII)}$$

The net result is that the *equation of a bicursal cubic Γ can be represented in the symbolic irrational form* (*VIII*), *provided that $S_{\lambda 1}$, $S_{\lambda 2}$, $S_{\lambda 3}$ are an arbitrary triad of cognate conics of double osculation,* (*belonging to any one of the nine systems*).

We may remark in passing that the conics ($S_{\lambda 1}$, $S_{\lambda 2}$, $S_{\lambda 3}$) of (VIII) may, as a special case, consist of

(i) the three *cognate* sextactic conics included in the system $\{S_\lambda\}$

or (ii) *any three* of the four pairs of tangents

$$(\xi_1, \eta_1), (\xi_2, \eta_2), (\xi_3, \eta_3), \text{ and } (\xi_4, \eta_4),$$

drawn respectively to Γ at the four pairs of inflexions, collinear with the (inflexion) O (Art. 2).

Finally, let us select (at random) two cognate conics of double osculation $S_{\lambda 1}$, $S_{\lambda 2}$ and then describe an *arbitrary* conic U through their (four) points of intersection. So we may write

$$U \equiv S_{\lambda 1} + k.\ S_{\lambda 2},$$

where k is an arbitrary parameter.

There is no difficulty in re-writing the equation (III) of Γ in the modified form:

$$(1+k)\, x^3 + y\, [3\, (\lambda_1 + k\lambda_2)\, x^2 + 3\, (\lambda_1{}^2 + k\lambda_2{}^2)\, xy + (\lambda_1{}^3 + k\lambda_2{}^3)\, y^2 - U] = 0.$$

Interpreting this equation geometrically, we come to conclude that, *if $S_{\lambda 1}$, and $S_{\lambda 2}$ be any two cognate conics of double osculation of a given bicursal cubic Γ, an arbitrary conic U, drawn through the four points of intersection* (*real or imaginary*) *of $S_{\lambda 1}$, $S_{\lambda 2}$ must cut Γ at six points, collinear in pairs with the related point of inflexion.*

In the next article we shall refer to certain loci and envelopes, connected with a family of cognate conics of double osculation, viz., $\{S_\lambda\}$.

Art. 4. Scrutinising the *point-equation* of $\{S_\lambda\}$ as well as its *tangential* equation, one can easily verify that Chasles' characteristics (μ, v) for the system are given by

$$\mu = 3 \quad \text{and} \quad v = 4.$$

This corroborates the results, (arrived at in Art. 2) viz., that $\{S_\lambda\}$ contains, within its fold, *four* parabolas and *three* rectangular hyperbolas.

Now the centre-locus (Ω) of $\{S_\lambda\}$—which can be anticipated to be a curve of degree v (i.e., 4)—is easily obtained in the form :

$$\begin{vmatrix} 1, & \frac{3}{2}x, & 0, & hx+by+f, & 0 \\ 0, & y, & \frac{3}{2}x, & 0, & hx+by+f \\ \frac{3}{2}, & 3x, & ax+hy+g, & 0, & 0 \\ 0, & \frac{3}{2}y, & 3x, & ax+hy+g, & 0 \\ 0, & 0, & \frac{3}{2}y, & 3x, & ax+hy+g \end{vmatrix} = 0.$$

By *a' priori* reasoning one can easily substantiate the following statements concerning the quartic Ω :—

(i) that the four asymptotes of Ω are parallel respectively to the axes of the four parabolas, included in the family $\{S_\lambda\}$;

and (ii) that the four points of intersection of Ω with the harmonic polar ζ (of O), viz.,

$$\zeta \equiv gx+fy+c=0$$

are precisely the points of intersection of the four pairs of inflexional tangents, symbolised as

(ξ_1, η_1), (ξ_2, η_2), (ξ_3, η_3) and (ξ_4, η_4) in Art. 2.

Next the envelope of the asymptotes of $\{S_\lambda\}$ can, without much difficulty, be shewn to be a curve of the 14th class, touching the eight inflexional tangents (just mentioned) and having the line at infinity for a multiple tangent.

Furthermore the locus of the foci of $\{S_\lambda\}$ can be identified as a curve of the 12th degree, *confocal with the original cubic Γ*.

Interested readers may, at their discretion, investigate other loci and envelopes, intrinsically related to a family of conics of double osculation.

SECTION II

(Invariants, covariants and contravariants of special triads of cognate conics of double osculation)

Art. 5.—Starting with a bicursal cubic Γ, let us, as before, take one of the nine points of inflexion as the origin O of Cartesian axes, and represent (after the manner of Art. 2) the associated family of conics of double osculation in the compact form:

$$\left.\begin{array}{l} S\lambda_r \equiv S+3\lambda_r x^2+3\lambda_r^2 . \, xy+\lambda_r^3 . y^2=0, \\ \text{i.e., } (a+3\lambda_r)x^2+2(h+\tfrac{3}{2}\lambda_r^2)xy+(b+\lambda_r^3)y^2+2gx+2fy+c=0, \end{array}\right\} \quad \ldots \quad (\mathrm{I})$$

it being postulated that λ_r is a variable parameter.

Let us now select an *arbitrary* triad of conics of the system, viz.,

$$(S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3}) \quad \ldots \quad (\mathrm{II})$$

by ascribing *arbitrary* values λ_1, λ_2, λ_3 to the parameter.

In connection with (II), we shall introduce the three notations :

(a) $M\,(S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3},)$,

(b) $J\,(S_{\lambda 1}\, S_{\lambda 2}, S_{\lambda 3})$,

and (c) $\phi\,(S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3})$

to denote respectively the M-invariant[1], the Jacobian covariant[2] and the cubic contravariant[3].

1. The M-invariant of three ternary quadratics U, V, W is the simplest rational function of the co-efficients, which, equated to zero, determines the condition for the expression

$$p_1 U+p_2 V+p_3 W$$

to be rendered a perfect square by an appropriate choice of the constants p_1, p_2, p_3.

2. As is well-known, the Jacobian-covariant is the locus of a point, whose polars *w. r. t.* the three conics are concurrent.

3. The cubic contravariant of three conics is the simplest function of the line-

Deferring for the present the discussion of (b) and (c) let us now concentrate our attention upon (a).

Employing the standard method of Higher Algebra, we can verify without much trouble that the M-invariant, attaching to (II), is identical (save as to a multiplicative constant) with the determinant

$$\begin{vmatrix} A+c\lambda_1^3, & A+c\lambda_2^3, & A+c\lambda_3^3 \\ B+3c\lambda_1, & B+3c\lambda_2, & B+3c\lambda_3 \\ H-\tfrac{3}{2}c\lambda_1^2, & H-\tfrac{3}{2}c\lambda_2^2, & H-\tfrac{3}{2}c\lambda_3^2 \end{vmatrix},$$

provided, of course, that the redundant factor

$$(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)(\lambda_1-\lambda_2)$$

is omitted.

Accordingly we may in all fairness set

$$M(S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3}) = \begin{vmatrix} A+c\lambda_1^3, & A+c\lambda_2^3, & A+c\lambda_3^3 \\ B+3c\lambda_1, & B+3c\lambda_2, & B+3c\lambda_3 \\ H-\tfrac{3}{2}c\lambda_1^2, & H-\tfrac{3}{2}c\lambda_2^2, & H-\tfrac{3}{2}c\lambda_3^2 \end{vmatrix} \div \tfrac{3}{2}c^2 \begin{vmatrix} \lambda_1^2, \lambda_2^2, \lambda_3^2 \\ \lambda_1, \lambda_2, \lambda_3 \\ 1, 1, 1 \end{vmatrix} \quad \ldots\text{(III)}$$

$$= 3A + 2H(\lambda_1+\lambda_2+\lambda_3) + B(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2) + 3c\lambda_1\lambda_2\lambda_3.$$

Our next objective is to characterise geometrically the triad of conics (II), qualified by the condition:

$$\left.\begin{aligned} &M(S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3}) = 0, \\ \text{i.e.,}\quad &3A+2H(\lambda_1+\lambda_2+\lambda_3)+B(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2)+3c\lambda_1\lambda_2\lambda_3 = 0. \end{aligned}\right\} \ldots\text{(IV)}$$

To that end it is necessary to attend to the following obvious facts:—

(i) that the harmonic polar (ζ) of the inflexion O, viz.,

$$\zeta \equiv gx+fy+c=0$$

represents the *common* polar line of O *w.r.t.* all conics of the system $\{S_{\lambda_r}\}$—not excepting S_{λ_1}, S_{λ_2}, S_{λ_3};

and (ii) that the equation of the pair of tangents, drawn from O to S_{λ_r} is

$$(B+3c\lambda_r)x^2 - 2(H-\tfrac{3}{2}c\lambda_r^2)xy + (A+c\lambda_r^3)y^2 = 0. \quad \ldots \quad \ldots \quad \text{(V)}$$

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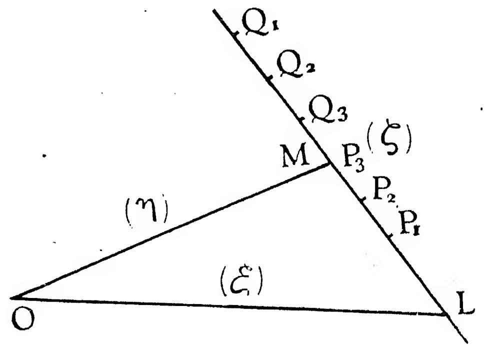
coordinates (l, m, n), which must vanish in order that the line may cut the conics in pairs of points in involution.. (Cf. Salmon's *Conic Sections* (1911), Arts. 388 (b) and 389 (a).

Suppose now (as shewn in the annexed figure) that the harmonic polar (ζ) intersects the three conics $S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3}$ respectively at the three point-pairs: $(P_1, Q_1), (P_2, Q_2), (P_3, Q_3)$. ... ... (VI)



Then, because of (i), the three line-pairs viz.,

$$(OP_1, OQ_1), \ (OP_2, OQ_2) \text{ and } (OP_3, OQ_3) \quad \dots \dots \dots \quad \text{(VII)}$$

are respectively the three pairs of tangents, drawn from 0 to $S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3}$. Hence by (ii), the three line-pairs are represented by the Cartesian equation (V), viz.,

$$(B+3\,c\lambda_r)\,x^2-2\,(H-\tfrac{3}{2}c\,\lambda_r^2)\,xy+(A+c\lambda_r^3)\,y^2=0, \quad \dots \dots \quad \text{(VIII)}$$

provided that r is allowed to run through the values 1, 2, 3.

It is now a pleasant job to verify that the assumed relation (IV)—which is the same as

$$\begin{vmatrix} A+c\lambda_1^3\,, & A+c\lambda_2^3\,, & A+c\lambda_3^3 \\ B+3c\lambda_1\,, & B+3c\lambda_2\,, & B+3c\lambda_3 \\ H-\tfrac{3}{2}c\lambda_1^2\,, & H-\tfrac{3}{2}c\lambda_2^2\,, & H-\tfrac{3}{2}c\lambda_3^2 \end{vmatrix}=0 \quad \dots \dots \quad \text{(IX)}$$

signifies geometrically that the three line-pairs (VII), defined analytically by (VIII), belong to the *same* involution,—or, what is the same thing—that the three point-pairs (VI) make up an involution. If, then, (L, M) be the two *focal points* of the point-involution (VI), the lines (OL, OM) must be the two *focal lines* of the line-involution (VII).

Furthermore the three point-pairs (P_1, Q_1), (P_2, Q_2) and (P_3, Q_3) being each harmonically conjugate with the two foci (L, M), it follows that the polars of the three points:

$$P_1, P_2, P_3,$$

with respect to *any one* of the three conics $S_{\lambda 1}$, $S_{\lambda 2}$, $S_{\lambda 3}$ pass respectively through the three points:

$$Q_1, Q_2, Q_3,$$

and *vice versa.* If we now pay heed to (i), we are led to infer that the triangle OLM is *self-conjugate* with respect to each of the three conics $S_{\lambda 1}$, $S_{\lambda 2}$, $S_{\lambda 3}$. Thus the relation (IV) or (IX) affirms the existence of a triangle, self-conjugate to all the three conics.

Further reference to Analytical Projective Geometry makes it clear that, subject to any of the equivalent relations (IV), (IX), the two focal lines (ξ, η)—i. e., (OL, OM)—of the line-involution formed by (VII) are representable in the Cartesian form:

$$3\xi\eta \equiv 3x^2+2(\lambda_1+\lambda_2+\lambda_3)xy+(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2)y^2=0. \quad \ldots \quad \ldots \quad \ldots \text{(X)}$$

A more *direct* method of confirming this result is to verify (by Elementary Analytical Geometry) that the *expanded* form of the relation

$$M(S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3})=0,$$

viz., $3A+2H(\lambda_1+\lambda_2+\lambda_3)+B(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2)+3c\,\lambda_1\lambda_2\lambda_3=0$ is interpretable as the necessary and sufficient condition that each of the three line-pairs (VII) represented by (VIII) may be harmonically conjugate with the line-pair (X).

The geometrical interpretation of (IV) being thus almost complete, we may summarise our conclusions as under :—

The invariant relation :

$$M(S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3})=0$$

represents the necessary and sufficient condition for the three conics of double osculation viz.,* $S_{\lambda 1}$, $S_{\lambda 2}$, $S_{\lambda 3}$ *to possess a common self-conjugate triangle* Δ.

* We may remark incidentally that the possession of a *common* self-conjugate triangle by three conics implies, in general, *three* distinct conditions, whereas the vanishing of their M-invariant amounts to a *single* condition. So it stands to reason

Further the aforesaid condition being fulfilled, the sides (ξ, η, ζ) *of the triangle* Δ *are given by the Cartesian equations:*

$$\zeta \equiv gx+fy+c=0$$

and $$3\,\xi\eta \equiv 3x^2+2\,(\lambda_1+\lambda_2+\lambda_3)\,xy+(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2)\,y^2=0.$$

Before we finish this article we shall utilise a familiar lemma of Analytical Geometry that the Jacobian of three conics, possessing a common self-conjugate triangle Δ, is a *degenerate* cubic, composed of three right lines, which are none other than the sides of Δ. Applying this lemma in the above context we are at once driven to the conclusion that, when the M-invariant of $S_{\lambda 1}$, $S_{\lambda 2}$, $S_{\lambda 3}$ vanishes, the three conics must have a common self-conjugate triangle, whose sides (ξ, η, ζ), taken together, constitute the degenerate Jacobian curve J $(S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3})$.

A *direct* proof of this result will be one among many items of business to be disposed of in the next article.

Art. 6.—We shall now deal with the Jacobian curve J$(S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3})$, mentioned in (*b*) of Art. 5. By the traditional method of Differential Calculus, the Jacobian of three *arbitrary* conics (of double osculation)

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that the M-invariant vanishes for a triad of conics endowed with a *common* self-conjugate triangle, but that the converse is *not* true under *normal* circumstances; in other words, three conics, having a *zero* M-invariant, may or may not have a common self-conjugate triangle. In fact, the possession of a common self-conjugate triangle by a triad of conics, qualified by a *zero* M-invariant, must be regarded as *accidental* rather than *natural*.

It is hardly necessary to point out that the triad of conics, considered as above, belongs to a *special* category. As a matter of fact *two* out of the three conditions—ordinarily needed to ensure the existence of a self-polar triangle—have been automatically fulfilled by $S_{\lambda 1}$, $S_{\lambda 2}$, $S_{\lambda 3}$, because of their being *cognate* conics of double osculation of a cubic. The *third* condition, which has yet to be fulfilled and which is practically the only condition that counts, is

$$M\,(S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3})=0.$$

This explains why and how this single condition is not only necessary but also sufficient for the three conics to have a common self-conjugate triangle.

viz., S_{λ_1}, S_{λ_2}, S_{λ_3} is readily obtained in the form :

$$\begin{vmatrix} (a+3\lambda_1)x+(h+\frac{3}{2}\lambda_1^2)y+g, & (h+\frac{3}{2}\lambda_1^2)x+(b+\lambda_1^3)y+f, & gx+fy+c \\ (a+3\lambda_2)x+(h+\frac{3}{2}\lambda_2^2)y+g, & (h+\frac{3}{2}\lambda_2^2)x+(b+\lambda_2^3)y+f, & gx+fy+c \\ (a+3\lambda_3)x+(h+\frac{3}{2}\lambda_3^2)y+g, & (h+\frac{3}{2}\lambda_3^2)x+(b+\lambda_3^3)y+f, & gx+fy+c \end{vmatrix}=0,$$

which can be put in the form :

$$(gx+fy+c)\times\left\{3x^2.\begin{vmatrix}1 & \lambda_1 & \lambda_1^2\\ 1 & \lambda_2 & \lambda_2^2\\ 1 & \lambda_3 & \lambda_3^2\end{vmatrix}+2xy\begin{vmatrix}1 & \lambda_1 & \lambda_1^3\\ 1 & \lambda_2 & \lambda_2^3\\ 1 & \lambda_3 & \lambda_3^3\end{vmatrix}+y^2\begin{vmatrix}1 & \lambda_1^2 & \lambda_1^3\\ 1 & \lambda_2^2 & \lambda_2^3\\ 1 & \lambda_3^2 & \lambda_3^3\end{vmatrix}\right\}=0,$$

i. ., $\quad (gx+fy+c)\left\{3x^2+2\,(\lambda_1+\lambda_2+\lambda_3)xy+(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2)y^2\right\}=0,$

i.e., $\quad \zeta\xi\eta=0,$

where the three lines ξ, η, ζ have the same significance as in Art. 5. That the harmonic polar ζ should form part of the Jacobian cubic is a foregone conclusion, considering that the line ζ is the common polar of a certain point (viz., O) *w. r. t.* the three conics S_{λ_1}, S_{λ_2}, S_{λ_3} and that reciprocally the polars of any point on ζ w.r.t. S_{λ_1}, S_{λ_2}, S_{λ_3} are concurrent, (the point of concurrence being O).

Thus the verification of the result, promised in the concluding portion of the previous article, is now complete ; nay, we have proved something more. For, the above result has nothing to do with the evanescence or non-evanescence of the M-invariant; in other words S_{λ_1}, S_{λ_2}, S_{λ_3} may be three *perfectly* arbitrary conics selected out of the system (S_{λ_r}). For the sake of clarity, we may finalise our result in the following manner :

No matter the M-invariant viz.,

$$M\,(S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3})$$

is or is not zero, the order-cubic viz.,

$$J\,(S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3})=0,$$

associated with any three cognate conics of double osculation viz., S_{λ_1}, S_{λ_2} S_{λ_3}, *is permanently an improper cubic curve, composed of the three right lines* ζ, ξ, η *defined by*

$$\zeta \equiv gx+fy+c=0$$

and $3\xi\eta \equiv 3x^2+2(\lambda_1+\lambda_2+\lambda_3)\, xy+(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2)y^2=0.$

The special feature, attaching to the triad of conics $(S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3})$, *restricted by the condition*

$$M\ (S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3})=0$$

is that the three constituent lines (ξ, η, ζ) *of the Jacobian make up a triangle, self-conjugate with respect to the three conics.*

As an illustration of the afore-mentioned proposition let us consider the triad of six-pointic conics, included in the system $\{S_\lambda\}$.

Referring to (V) of case i (Art. 2), we note that, if $S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3}$ denote the three sextactic conics of the family $\{S_\lambda\}$, $\lambda_1, \lambda_2, \lambda_3$ must be the three roots of the cubic:

$$c\lambda^3+B\lambda^2+2H\lambda+A=0,$$

so that

$$\lambda_1+\lambda_2+\lambda_3=-\frac{B}{c}, \qquad \lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2=\frac{2H}{c} \qquad \text{and} \qquad \lambda_1\lambda_2\lambda_3=-\frac{A}{c}.$$

Hence, by (III) of Art. 5,

$$M\ (S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3})=3A+2H\times\left(-\frac{B}{c}\right)+B\times\frac{2H}{c}+3c\left(-\frac{A}{c}\right)=0.$$

Also the joint equation of (ξ, η),—viz., (X) of Art. 5—simplifies to

$$3x^2+2\left(-\frac{B}{c}\right)xy+\frac{2H}{c}y^2=0,$$

i. e.,
$$3cx^2-2\,Bxy+2Hy^2=0.$$

Joining this special result to the other more general results proved heretofore, we can formulate our final conclusions in the following garb:—

The necessary and sufficient condition for three cognate conics of double osculation viz., $(S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3})$ *to claim a common self-conjugate triangle is*

that their M-invariant should vanish, i. e., that the parametres λ_1, λ_2, λ_3, *should conform to the relation:*

$$3A+2H(\lambda_1+\lambda_2+\lambda_3)+B(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2)+3c\lambda_1\lambda_2\lambda_3=0.$$

Subject to this condition, the sides of the common self-conjugate triangle are simply the three right lines (ξ, η, ζ) *which constitute the degenerate Jacobian curve*

$$J\,(S_{\lambda_1}, S_{\lambda_2}, S_{\lambda_3})=0,$$

so that the equations of the three lines are

$$\zeta\equiv gx+fy+c=0$$

and $$3\xi\eta\equiv 3x^2+2(\lambda_1+\lambda_2+\lambda_3)\,xy+(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2)\,y^2=0.$$

A remarkably special triad of conics of the above description is that composed of the three sextactic conics, which belong to the system and whose parameters λ_1, λ_2, λ_3 *are the three roots of the cubic in* λ, *viz.,*

$$c\lambda^3+B\lambda^2+2H\lambda+A=0.$$

Furthermore the common self-conjugate triangle of these three (sextactic) conics is formed by the harmonic polar:

$$\zeta\equiv gx+fy+c=0,$$

and the line-pair (ξ, η) *given by*

$$3c\xi\eta\equiv 3cx^2-2Bxy+2Hy$$

Art. 7.—We shall now offer general criticism on an *arbitrary* triad of conics of double osculation.

General reasoning shews that *four* conditions have to be satisfied in order that a conic may have *double osculation* with a cubic. So in order that three *given* conics U, V, W may each have double osculation with an *undefined* cubic (say, Σ), 3×4 or 12 conditions –involving the *known* constants of U, V, W and the nine *unknown or disposable* constants occurring in the analytic structure of Σ—must be fulfilled. So one can readily surmise that, *in order that three given conics U. V. W may be cognate conics of double osculation of a certain cubic* (12 –9, *i.e.*,3) *three conditions must be satisfied by the conics.* This is quite in consonance with the result of the previous article, considering that the reduction of the (cubic) Jacobian to a triad of right lines is tantamount to *three* condi-

tions, and that a necessary condition for U, V, W to be cognate conics of double osculation is that their Jacobian should break up into three right lines. But in as much as counting of constants or of conditions is not always a safe process, we have no reason to affirm anything positively regarding the *converse* case, which accordingly calls for a special scrutiny. Inquisitive students may propose to examine how far (if at all) the converse proposition is valid.

Art. 8.—We shall now conclude this section with a brief reference to the cubic contravariant :

$$\phi(S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3}),$$

—mentioned at the beginning of Art. 5,—whose vanishing expresses the criterion that the right line, given by the *line-coordinates* (l, m, n) —and therefore by the Cartesian equation :

$$lx+my+n=0$$

—may intersect the three conics of double osculation viz., $S_{\lambda 1}$, $S_{\lambda 2}$, $S_{\lambda 3}$ in pairs of points in involution.

By the prescribed method of Analytical Geometry the condition in question is easily obtained in the form :

$$\begin{vmatrix} P+3\lambda_1 n^2, & Q+\lambda_1^3 n^2, & R+\frac{3}{2}\lambda_1^2 n^2 \\ P+3\lambda_2 n^2, & Q+\lambda_2^3 n^2, & R+\frac{3}{2}\lambda_2^2 n^2 \\ P+3\lambda_3 n^2, & Q+\lambda_3^3 n^2, & R+\frac{3}{2}\lambda_3^2 n^2 \end{vmatrix}=0, \ldots \qquad \text{(I)}$$

where

$$\left.\begin{aligned} P&\equiv cl^2-2\,gln+an^2, \\ Q&\equiv cm^2-2fmn+bn^2, \\ \text{and} \quad R&\equiv clm-n(fl+gm)+hn^2. \end{aligned}\right\} \quad \ldots \ \ldots \ \ldots \ \ldots \qquad \text{(II)}$$

By easy manipulations and reductions, (I) can be developed into the form :

$$n\left[3Q-2R(\lambda_1+\lambda_2+\lambda_3)+P(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2)+3n^2\lambda_1\lambda_2\lambda_3\right]=0.$$

The immediate inference is that the class-cubic, which normally envelopes the set of ∞^1 right lines, cutting an arbitrarily assigned triad of conics in pairs of points in involution, has, in the case of the special triad $(S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3})$, broken up into a curve of class *zero* viz., the ori-

gin ($n=0$), and a class-conicΣ, determined by the tangential equation:

$$3\lambda_1\lambda_2\lambda_3 .\ n^2+(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2).\ (cl^2-2gln+an^2)$$
$$-2\ (\lambda_1+\lambda_2+\lambda_3)\ (clm-fnl-gmn+hn^2)+3\ (cm^2-2fmn+bn^2)=0. \quad \text{(III)}$$

An important corollary is that *every* right line, drawn through the curve of class zero (viz., the inflexion O) cuts the series of conics $\{ S_{\lambda r} \}$ in pairs of points in involution,—a result, admitting of independent verification.

For the triad of conics ($S_{\lambda 1}$, $S_{\lambda 2}$, $S_{\lambda 3}$), qualified by the condition:

$$M\ (S_{\lambda 1},\ S_{\lambda 2},\ S_{\lambda 3})=0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(IV)}$$

the class-conic Σ can be easily shewn to reduce to a pair of points, so that the class-cubic, defined by the cubic contravariant, is compounded of three *points* (i.e., three curves of class zero). A moment's reflection leads to the conclusion that the triangle, formed by these points, is none other than the common self-conjugate triangle, proved to exist for the triad of conics under the condition (IV). (See Art. 5).

Section III

(Canonical forms of triads of conics $S_{\lambda 1}$, $S_{\lambda 2}$, $S_{\lambda 3}$, having a zero M-invariant)

Art. 9.—Suppose as before that $S_{\lambda 1}$, $S_{\lambda 2}$, and $S_{\lambda 3}$, are three cognate conics of double osculation, having a zero M-invariant and therefore conforming to the relations:

$$M(S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3})\equiv 3A+2H(\lambda_1+\lambda+\lambda_3)+B\,(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2)+3c\ \lambda_1\lambda_2\lambda_3$$
$$=0. \quad \ldots \quad \ldots \quad \text{(I)}$$

Then, as shewn in Art. 6, the three conics must have a common self-conjugate triangle, whose sides ζ, η, ξ are given by

$$\zeta\equiv gx+fy+c=0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

and $3\xi\eta\equiv 3x^2+2(\lambda_1+\lambda_2+\lambda_3)\ xy+(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2)\,y^2=0.$ ... ... (2)

Manifestly, then, if the individual equations of the lines ξ, η (passing through O) be written as

$$\xi\equiv x-\mu y=0 \quad \text{and} \quad \eta\equiv x-\nu y=0, \quad \ldots \quad \ldots \quad (3)$$

then μ,ν must be the two roots of the quadratic in t, viz.

$$3t^2+2(\lambda_1+\lambda_2+\lambda_3)\ t+(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2)=0, \quad \ldots \quad \ldots \quad (4)$$

so that

$$\left.\begin{array}{l} \mu+\nu=-\tfrac{2}{3}\,(\lambda_1+\lambda_2+\lambda_3), \\ \text{and} \quad \mu\nu=\ \tfrac{1}{3}\,(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2). \; \ldots \quad \ldots \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Recollecting the familiar proposition on a triad of conics, endowed with a common self-conjugate triangle, we infer at once that there must exist nine numerical constants viz.,

$$l_1,\ m_1,\ n_1,\ l_2,\ m_2,\ n_2,\ l_3,\ m_3,\ n_3,$$

compatible with the three *identities* (in x, y), viz.

$$S_{\lambda 1} \equiv l_1\xi^2+m_1\eta^2+n_1\zeta^2, \quad \ldots \quad \ldots \quad (6)$$

$$S_{\lambda 2} \equiv l_2\xi^2+m_2\eta^2+n_2\zeta^2, \quad \ldots \quad \ldots \quad (7)$$

$$S_{\lambda 3} \equiv l_3\xi^2+m_3\eta^2+n_3\zeta^2. \quad \ldots \quad \ldots \quad (8)$$

Replacing $S_{\lambda 1}$, $S_{\lambda 2}$, $S_{\lambda 3}$ by their equivalent values (in terms of x, y), viz., three expressions of the type

$$S_{\lambda_r} \equiv (a+3\lambda_r)x^2+2(h+\tfrac{3}{2}\lambda_r^2)xy+(b+\lambda_r^3)y^2+2gx+2fy+c, \; (r=1,2,3), \quad (9)$$

and substituting for ξ, η, ζ the functional values (in terms of x, y) as provided for by (1) and (3), we can readily convert (6), (7), (8) respectively into three identities of the type:

$$(a+3\lambda_r)x^2+2(h+\tfrac{3}{2}\lambda_r^2)xy+(b+\lambda_r^3)\,y^2+2gx+2fy+c$$

$$\equiv l_r\,(x-\mu y)^2+m_r(x-\nu y)^2+n_r\,(gx+fy+c)^2, \quad (r=1,\ 2,\ 3). \quad \ldots \quad (10)$$

Equating co-efficients of like terms in x, y, we derive relations of the type:

$$n_r = \frac{1}{c}, \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$l_r+m_r=3\lambda_r+\frac{B}{c}, \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

$$l_r\mu+m_r\nu=\frac{H}{c}-\frac{3}{2}\lambda_r^2, \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

and

$$l_r\mu^2+m_r\nu^2=\frac{A}{c}+\lambda_r^3, \quad \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

(r being as usual allowed to run through the values 1, 2, 3). Obviously, the equation (i) gives n_r directly, and *any* two of the other three equations viz., (ii), (iii), (iv) can be solved linearly for l_r, m_r.

Needless to say, the mutual consistency of (ii), (iii), (iv) follows from the fact that the determinant:

$$\begin{vmatrix} 1 , & \mu , & \mu^2 \\ 1 , & \nu , & \nu^2 \\ 3\lambda_r+\frac{B}{c}, & \frac{H}{c}-\frac{3}{2}\lambda_r^2, & \frac{A}{c}+\lambda_r^3 \end{vmatrix}$$

vanishes by virtue of (I) and (5). This is as it should be, for the coexistence of the canonical forms (6), (7), (8) for $S_{\lambda1}$, $S_{\lambda2}$, $S_{\lambda3}$ is a direct consequence of their having a common self-conjugate triangle (viz., ξ, η, ζ).

Now reverting to our former topic, and solving (ii), (iii) for l_r, m_r we derive

$$\left.\begin{aligned} l_r &= \frac{\frac{H}{c}-\frac{3}{2}\lambda_r^2-(3\lambda_r+\frac{B}{c})\nu}{\mu-\nu}, \\ \text{and} \quad m_r &= \frac{(3\lambda_r+\frac{B}{c})\mu-(\frac{H}{c}-\frac{3}{2}\lambda_r^2)}{\mu-\nu}. \end{aligned}\right| (r=1, 2, 3). \quad \ldots \quad \ldots \quad \text{(V)}$$

Joining the main results of this article to the proved results of the previous articles, we can summarise our conclusions in the undermentioned form:

The invariant relation

$$M\,(S_{\lambda1}, S_{\lambda2}, S_{\lambda3})=0 \quad \ldots \quad \ldots \quad \text{(I)}$$

represents the necessary and sufficient condition that the three cognate conics of double osculation $S_{\lambda1}$, $S_{\lambda2}$, $S_{\lambda3}$ *may admit of the simultaneous canonical forms :*

$$\left.\begin{aligned} S_{\lambda1} &= l_1\,\xi^2+m_1\,\eta^2+n_1\,\zeta^2, \\ S_{\lambda2} &= l_2\,\xi^2+m_2\,\eta^2+n_2\,\zeta^2, \\ S_{\lambda3} &= l_3\,\xi^2+m_3\,\eta^2+n_3\,\zeta^2, \end{aligned}\right\}$$

where (ξ, η, ζ) *are three determinate right lines and* $\{l_r\}$, $\{m_r\}$ *and* $\{n_r\}$ *are determinate constants. Looked at from a geometrical standpoint, the three lines*

ξ, η, ζ form the common self-conjugate triangle that the three conics must possess under the condition (I), *whereas the improper cubic curve made up of the three lines is none other than their Jacobian. The actual Cartesian equations to the three lines are*

$$\xi \equiv x-\mu y=0, \qquad \eta \equiv x-\nu y=0 \qquad \zeta \equiv gx+fy+c=0$$

where μ, ν *are the two roots of the quadratic in* t, *viz.*

$$3t^2+2\,(\lambda_1+\lambda_2+\lambda_3)t+(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2)=0.$$

Furthermore the coefficients

$$\{l_r\}, \quad \{m_r\} \quad \text{and} \quad \{n_r\}$$

are given by

$$\left.\begin{aligned} l_r &= \frac{\frac{H}{c}-\frac{3}{2}\lambda_r^2-\left(3\lambda_r+\frac{B}{c}\right)\nu}{\mu-\nu}, \\ m_r &= \frac{\left(3\lambda_r+\frac{B}{c}\right)\mu-\left(\frac{H}{c}-\frac{3}{2}\lambda_r^2\right)}{\mu-\nu}, \\ \text{and} \quad n_r &= \frac{1}{c}. \end{aligned}\right\}$$

N. B.—It is hardly necessary to add that the third coefficient of the type $\{n_r\}$ is independent of r, being, as it is, equal to $\frac{1}{c}$. So $n_1=n_2=n_3=\frac{1}{c}$; still for the sake of symmetry the notations n_1, n_2, n_3 have been retained.

Art. 10.—As an illustration of the general proposition of the preceding article, let us consider the triad of cognate *sextactic* conics S_{λ_1}, S_{λ_2}, S_{λ_3}, whose parameters are known to conform to the relations:

$$\left.\begin{aligned} \lambda_1+\lambda_2+\lambda_3 &= -\frac{B}{c}, \\ \lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2 &= \frac{2H}{c} \\ \text{and} \quad \lambda_1\lambda_2\lambda_3 &= -\frac{A}{c}. \end{aligned}\right\}$$

Now by (4) and (5) of Art. 9 we see that μ, ν are the roots of the

quadratic in t, viz.

$$3ct^2-2Bt+2H=0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

so that

$$\mu+\nu=\frac{2B}{3c} \quad \text{and} \quad \mu\nu=\frac{2H}{3c}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Hence (V) of Art. 9 gives

$$l_r=\frac{3}{2(\nu-\mu)}\left(\lambda_r^2+2\lambda_r\nu+\frac{2B}{3c}\cdot\nu-\frac{2H}{3c}\right)$$

$$=\frac{3}{2(\nu-\mu)}\cdot\left\{(\lambda_r+\nu)^2-\left(\nu^2-\frac{2B}{3c}\cdot\nu+\frac{2H}{3c}\right)\right\} \qquad \ldots \quad \ldots \quad (3)$$

and

$$m_r=\frac{3}{2(\mu-\nu)}\cdot\left(\lambda_r^2+2\lambda_r\mu+\frac{2B}{3c}\cdot\mu-\frac{2H}{3c}\right)$$

$$=\frac{3}{2(\mu-\nu)}\cdot\left\{(\lambda_r+\mu)^2-\left(\mu^2-\frac{2B}{3c}\cdot\mu+\frac{2H}{3c}\right)\right\}. \qquad \ldots \quad (4)$$

Now by (2)

$$\nu^2-\frac{2B}{3c}\cdot\nu+\frac{2H}{3c}=\nu^2-\nu(\mu+\nu)+\mu\nu=0,$$

and

$$\mu^2-\frac{2B}{3c}\cdot\mu+\frac{2H}{3c}=\mu^2-\mu(\mu+\nu)+\mu\nu=0.$$

These relations follow also from the consideration that μ, ν are the roots of (1).

Hence (3) and (4) simplify to

$$l_r=\frac{3(\lambda_r+\nu)^2}{2(\nu-\mu)},$$

and

$$m_r=\frac{3(\lambda_r+\mu)^2}{2(\mu-\nu)}.$$

We may then sum up our results as under :—

The three cognate sextactic conics S_{λ_1}, S_{λ_2}, S_{λ_3} belonging to the system $\{S\lambda_r\}$ can be simultaneously represented in the three canonical forms:

$$S_{\lambda_r} \equiv l_r \xi^2 + m_r \eta^2 + n_r \zeta^2, \qquad (r=1, 2, 3)$$

where the constants

$$\{l_r\}, \{m_r\}, \{n_r\}$$

are determined by

$$\left.\begin{aligned} l_r &= \frac{3(\lambda_r+\nu)^2}{2(\nu-\mu)}, \\ m_r &= \frac{3(\lambda_r+\mu)^2}{2(\mu-\nu)}, \\ \text{and} \qquad n_r &= \frac{1}{c}, \end{aligned}\right| \quad (r=1, 2, 3)$$

and the three right lines ξ, η, ζ *are defined by*

$$\xi \equiv x-\mu y=0, \quad \eta \equiv x-\nu y=0 \quad \text{and } \zeta \equiv gx+y+c=0,$$

it being understood that μ, ν *are the two roots of the quadratic in* t, *viz.*

$$3ct^2-2Bt+2H=0.$$

SECTION IV

1. *Digression on a triad of conics with a common self-conjugate triangle*
2. *Application to a triad of cognate conics* S_{λ_1}, S_{λ_2}, S_{λ_3}, *of zero M-invariant*

Art. 11—We know from the Theory of Higher Plane Curves that, *in general*, a uniquely determinate cubic can be found so as to have three arbitrarily assigned conics U, V, W for polar conics. Let us now proceed to examine how far this theorem is valid when the three conics happen to possess a common self-conjugate triangle (say, Δ). Then on taking Δ as the fundamental triangle of reference, we may express the *homogeneous* equations of the three conics in the respective forms:

$$U \equiv a_1\xi^2+b_1\eta^2+c_1\zeta^2=0, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$V \equiv a_2\xi^2+b_2\eta^2+c_2\zeta^2=0, \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$\text{and} \quad W \equiv a_3\xi^2+b_3\eta^2+c_3\zeta^2=0. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Let Σ represent the cubic, having for its homogeneous equation:

$$l\xi^3+m\eta^3+n\zeta^3=0, \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

where the parametric constants $l:m:n$ may be arbitrarily chosen.

There is no difficulty in shewing that, for any particular set of values for $l:m:n$, the Hessian of Σ is a *degenerate* cubic curve composed of three right lines (viz., the three sides ξ, η, ζ of Δ), and that the pencil of (four) tangents, drawn to Σ from an *arbitrary* point on its periphery, is *equi-anharmonic*. In point of fact, Σ belongs to a wider class of cubic curves called '*equi-anharmonic*' by Hilton.*

If now P, Q, R denote respectively the three points, whose homogeneous (i.e., projective) coordinates are

$$\left(\frac{a_1}{l}, \frac{b_1}{m}, \frac{c_1}{n}\right), \quad \left(\frac{a_2}{l}, \frac{b_2}{m}, \frac{c_2}{n}\right) \text{ and } \left(\frac{a_3}{l}, \frac{b_3}{m}, \frac{c_3}{n}\right),$$

it is easy to see that the polar conics of P, Q, R w. r. t. the cubic Σ are respectively identical with the three original conics U, V, W defined by (1), (2), (3).

If, for a *given* triad of conics U, V, W, we go on varying the ratios $l:m:n$, the cubic Σ and the triad of points P, Q, R will also vary, but, then, the geometrical property, established as above, will continue to hold good.

Thus, whereas three assigned conics U, V, W are, *in general*, polar conics of three *determinate* points P, Q, R w. r. t. a *determinate* cubic curve Σ, the numbers of such cubics and also of the correlated triad of points (P, Q, R) will become *two-fold infinity* in the particular case, when U, V, W have a common self-conjugate triangle. †To be precise, *three conics having a common self-conjugate triangle ABC can be designated as polar conics, belonging to any one of a family of* ∞^2 *equi-anharmonic cubics, which have, for their common Hessian, the degenerate cubic made up of the three lines* (BC, CA, AB).

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*See Hilton's "*Plane Algebraic Curves*" (1920), P. 238, Ex. 6.

†Regard being had to the fact that the M-invariant vanishes for the triad of conics (U, V, W), possessing a common self-conjugate triangle, it is crystal-clear that Dr. Salmon's remark ["Conic Sections" (1911), Art. 389 (c)] viz., "*If the invariant M vanishes, an exception occurs and the conics cannot all be derived from the same cubic,*" does not seem to fit in with established facts.

Art. 12.—We can now ıeadily apply the results of the foregoing article to a triad of conics of double osculation $(S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3})$, belonging to a given cubic and possessing a zero M-invariant.

As shewn in Sec. III, the triad of conics will then have a common self-conjugate triangle (say, Δ) and their *homogeneous* equations, referred to Δ, will assume the respective forms:

$$\left.\begin{aligned} S_{\lambda 1} &\equiv l_1\xi^2+m_1\eta^2+n_1\zeta^2=0, \\ S_{\lambda 2} &\equiv l_2\xi^2+m_2\eta^2+n_2\zeta^2=0, \\ \text{and} \qquad S_{\lambda 3} &\equiv l_3\xi^2+m_3\eta^2+n_3\zeta^2=0, \end{aligned}\right\} \quad \ldots \quad \ldots \quad \text{(I)}$$

the coefficients

$$(l_1, m_1, n_1, \quad l_2, m_2, n_2, \quad l_3, m_3, n_3)$$

being defined by (V) of Art. 9.

Let Σ denote the cubic

$$p\xi^3+q\eta^3+r\zeta^3=0, \qquad \ldots \quad \ldots \quad \text{(II)}$$

where the coefficients $p:q:r$ are arbitrary.

It is a tane affair to verify that the three conics $S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3}$ given by (I), are respectively the polar conics of the three points, whose homogeneous or projective coordinates are

$$\left(\frac{l_1}{p}, \frac{m_1}{q}, \frac{n_1}{r}\right), \quad \left(\frac{l_2}{p}, \frac{m_2}{q}, \frac{n_2}{r}\right) \quad \text{and} \quad \left(\frac{l_3}{p}, \frac{m_3}{q}, \frac{n_3}{r}\right).$$

with respect to the cubic Σ.

It is needless to point out that, when the parameters p, q, r, are allowed to vary, the equation (II) will define a family of ∞^2 equianharmonic cubics, any one of which will have the given conics $(S_{\lambda 1}, S_{\lambda 2}, S_{\lambda 3})$ foı a triad of polar conics.

Of course the above result holds good even when $S\lambda_1$, $S\lambda_2$, $S\lambda_3$, are a triad of cognate sextactic conics.

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